'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2023

CC9-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-I

(REVISED SYLLABUS 2023 / OLD SYLLABUS 2018)

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.

GROUP-A

Answer any *four* questions from the following 3×4 = 12

1. Let F be a field, then show that the groups $(F \setminus \{0\}, \cdot)$ and $(F, +)$ cannot be isomorphic. 3

2. Find a basis and dimension of the subspace S of the vector space $M_2(\mathbb{R})$ over $\mathbb{R}$, where $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + b = 0 \right\}$. 3

3. Find a generator for each of the ideals $4\mathbb{Z}+10\mathbb{Z}$ and $8\mathbb{Z} \cap 12\mathbb{Z}$ of $\mathbb{Z}$. 3

4. Is the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ a field? Justify your answer. 3

5. Let V be a vector space with $\{\alpha, \beta, \gamma\}$ as a basis. Prove that the set $\{\alpha+\beta+\gamma, \beta+\gamma, \gamma\}$ is also a basis of V. 3

6. Find a linear mapping $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that 3

$$\operatorname{Im} T = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}.$$

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. The matrix of a linear mapping $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ relative to the ordered basis $\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ of $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$. Find T. Also find the matrix of T relative to the standard basis of $\mathbb{R}^3$. 3+3

8. Find all the units in the ring $\mathbb{Z}_{10}$. Prove that these units form a cyclic group under multiplication. 2+4

9. State and prove third isomorphism theorem on rings. 2+4

10. Determine the linear mapping $T:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ which maps the basis vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) of $\mathbb{R}^3$ to the vectors (1, 1), (2, 3), (3, 2) respectively. Prove that T is onto but not one-one. 3+3

11.(a) Prove that every Boolean ring is commutative. Is the converse true? Justify your answer. A ring is Boolean if its every element is idempotent. 3

(b) Let S and T be two ideals of a ring R. Prove that $S \cup T$ is an ideal of R iff either $S \subseteq T$ or $T \subseteq S$. 3

12.(a) In $\mathbb{R}^2$, consider $\alpha = (3, 1)$ and $\beta = (2, -1)$. Determine $L\{\alpha, \beta\}$ and show that $L\{\alpha, \beta\} = \mathbb{R}^2$. 3

(b) Prove that the set S of all 2×2 symmetric matrices with real entries is a subspace of $M_2(\mathbb{R})$. 3

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Prove that in a commutative ring R with identity, a proper ideal P of R is a prime ideal iff R/P is an integral domain. Use this result to prove that $\langle x \rangle$ is a prime ideal of $\mathbb{Z}[x]$. 4+2

(b) Give an example of each of the following: 2+2

(i) An infinite ring R (which is not a field) with char $R = 2$.

(ii) An infinite field F with char $F = 3$.

Here char S denotes the characteristic of the ring S.

(c) Prove that the characteristic of a Boolean ring is 2. 2

14.(a) Consider the subring R of $M_2(\mathbb{Z})$; where $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} :\ a, b \in \mathbb{Z} \right\}$. 2+2+2+2

Let $\phi : R \longrightarrow \mathbb{Z}$ be a map, defined by $\phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) = a - b$ for all $\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in R$.

(i) Show that ϕ is a homomorphism.

(ii) Determine $\ker\phi$.

(iii) Show that $R/\ker\phi$ is isomorphic to $\mathbb{Z}$.

(iv) Is $\ker\phi$ a prime ideal of R? Justify.

(b) Let $(R, +, \cdot)$ be a ring where $(R, +)$ is a cyclic group. Prove that R is a commutative ring. Use this result to prove that the rings of order 2, 3, 5, 6, 7 are commutative rings. 2+2

15.(a) Show that T is non-singular and determine T^{-1} where $T:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is a linear mapping defined by: 3+3

$$T(x,\ y,\ z) = (x-y,\ x+2y,\ y+3z) \text{ for all } (x,\ y,\ z) \in \mathbb{R}^3.$$

(b) Suppose a linear mapping $T: V \to W$ maps the ordered basis $\{\alpha_1,\ \alpha_2,\ \alpha_3\}$ of $V(\mathbb{R})$ as 4

$$T(\alpha_1) = \beta_1,\ T(\alpha_2) = \beta_1 + \beta_2,\ T(\alpha_3) = \beta_1 + \beta_2 + \beta_3$$

where $\{\beta_1,\ \beta_2,\ \beta_3\}$ is an ordered basis of $W(\mathbb{R})$. Find the matrix of T^{-1} relative to the same chosen ordered bases.

(c) Prove that the vector space $\mathbb{R}$ over $\mathbb{Q}$ is infinite dimensional. 2

16.(a) Let $W_1 = L\{(1,\ -2,\ 1),\ (2,\ 3,\ 5)\}$ and $W_2 = L\{(1,\ -2,\ 0),\ (3,\ -3,\ 0)\}$ then show that W_1 and W_2 are subspaces of $\mathbb{R}^3$. Determine $\dim W_1, \dim W_2$ and $\dim(W_1 + W_2)$. 4+4

(b) Find the dimension of $\ker T$ where $T:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is the linear transformation, given by $T(x,\ y,\ z) = (x+z,\ y+z)$ for all $(x,\ y,\ z) \in \mathbb{R}^3$. 4

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